

Supersymmetric content of the Dirac and Duffin-Kemmer-Petiau equations

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Abstract

We study subsolutions of the Dirac and Duffin-Kemmer-Petiau equations described in our earlier papers. It is shown that subsolutions of the Duffin-Kemmer-Petiau equations and those of the Dirac equation obey the same Dirac equation with some built-in projection operator. This covariant equation can be referred to as supersymmetric since it has bosonic as well as fermionic degrees of freedom.

1 Introduction

Recently, subsolutions of the Duffin-Kemmer-Petiau (DKP) equations were found and it was shown that the subsolutions fulfill the appropriately projected Dirac equation [13, 14]. On the other hand, massive subsolutions of the Dirac equation were also found and studied [15]. In the present paper we demonstrate that subsolutions of the DKP equations and those of the Dirac equation obey the same Dirac equation with some built-in projection operator. This equation was shown to be covariant in our earlier paper [15]. We shall refer to this equation as supersymmetric since it has bosonic (spin 0 and 1) as well as fermionic (spin $\frac{1}{2}$) degrees of freedom.

Some of the results described below were derived earlier but are included for the sake of completeness. The paper is organized as follows. In Section 2 the Dirac as well as the DKP equations are described shortly. The DKP equations for $s = 0$ are written as a set of two 3×3 equations in Section 3 (the case $s = 1$ leads to analogous equations [13]) and it is shown that their solutions fulfill the Dirac equation. In Section 4 subsolutions of the Dirac equation are described. Then in Section 5 it is demonstrated that all these subsolutions obey the Dirac equation with built-in projection operator, referred henceforth as the supersymmetric equation. Finally, the supersymmetric equation is written in representation independent form (its covariance was verified in [15]).

2 Relativistic equations

In what follows tensor indices are denoted with Greek letters, $\mu = 0, 1, 2, 3$. We shall use the following convention for the Minkowski space-time metric tensor: $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and we shall always sum over repeated indices. Four-momentum operators are defined in natural units, $c = 1$, $\hbar = 1$, as $p^\mu = i\frac{\partial}{\partial x_\mu}$.

2.1 Dirac equation

The Dirac equation is a relativistic quantum mechanical wave equation formulated by Paul Dirac in 1928 providing a description of elementary spin- $\frac{1}{2}$ particles, such as electrons and quarks, consistent with both the principles of quantum mechanics and the theory of special relativity [1]. The Dirac equation is [2, 3, 4]:

$$\gamma^\mu p_\mu \Psi = m\Psi, \quad (1)$$

where m is the rest mass of the elementary particle. The γ 's are 4×4 anti-commuting Dirac matrices: $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}I$ where I is a unit matrix.

In the spinor representation of the Dirac matrices we have $\gamma^0 = \begin{pmatrix} \mathbf{0} & \sigma^0 \\ \sigma^0 & \mathbf{0} \end{pmatrix}$, $\gamma^j = \begin{pmatrix} \mathbf{0} & -\sigma^j \\ \sigma^j & \mathbf{0} \end{pmatrix}$, $j = 1, 2, 3$, $\gamma^5 = \begin{pmatrix} \sigma^0 & \mathbf{0} \\ \mathbf{0} & -\sigma^0 \end{pmatrix}$. The wave function is a bispinor, i.e. consists of 2 two-component spinors ξ, η : $\Psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$.

2.2 Duffin-Kemmer-Petiau equations

The DKP equations for spin 0 and 1 are written as:

$$\beta_\mu p^\mu \Psi = m\Psi, \quad (2)$$

with 5×5 and 10×10 matrices β^μ , respectively, which fulfill the following commutation relations [5, 6]:

$$\beta^\lambda \beta^\mu \beta^\nu + \beta^\nu \beta^\mu \beta^\lambda = g^{\lambda\mu} \beta^\nu + g^{\nu\mu} \beta^\lambda. \quad (3)$$

In the case of 5×5 (spin-0) representation of β^μ matrices Eq.(2) is equivalent to the following set of equations:

$$\left. \begin{aligned} p^\mu \psi &= m\psi^\mu \\ p_\nu \psi^\nu &= m\psi \end{aligned} \right\}, \quad (4)$$

if we define Ψ in (2) as:

$$\Psi = (\psi^\mu, \psi)^T = (\psi^0, \psi^1, \psi^2, \psi^3, \psi)^T, \quad (5)$$

where T denotes transposition of a matrix. Let us note that Eq.(4) can be obtained by factorizing second-order derivatives in the Klein-Gordon equation $p_\mu p^\mu \psi = m^2 \psi$.

In the case of 10×10 (spin-1) representation of matrices β^μ Eq.(2) reduces to:

$$\left. \begin{aligned} p^\mu \psi^\nu - p^\nu \psi^\mu &= m \psi^{\mu\nu} \\ p_\mu \psi^{\mu\nu} &= m \psi^\nu \end{aligned} \right\}, \quad (6)$$

with the following definition of Ψ in (2):

$$\Psi = (\psi^{\mu\nu}, \psi^\lambda)^T = (\psi^{01}, \psi^{02}, \psi^{03}, \psi^{23}, \psi^{31}, \psi^{12}, \psi^0, \psi^1, \psi^2, \psi^3)^T, \quad (7)$$

where ψ^λ are real and $\psi^{\mu\nu}$ are purely imaginary (in alternative formulation we have $-\partial^\mu \psi^\nu + \partial^\nu \psi^\mu = m \psi^{\mu\nu}$, $\partial_\mu \psi^{\mu\nu} = m \psi^\nu$, where $\psi^\lambda, \psi^{\mu\nu}$ are real). Because of antisymmetry of $\psi^{\mu\nu}$ we have $p_\nu \psi^\nu = 0$ what implies spin 1 condition. The set of equations (6) was first written by Proca [8] and in a different context by Lanczos [9]. More on the rich history of the formalism of Duffin, Kemmer and Petiau can be found in [10].

3 Splitting the spin-0 Duffin-Kemmer-Petiau equations

Four-vectors $\psi^\mu = (\psi^0, \psi)$ and spinors $\zeta^{A\dot{B}}$ are related by formula:

$$\zeta^{A\dot{B}} = (\sigma^0 \psi^0 + \sigma \cdot \psi)^{A\dot{B}} = \begin{pmatrix} \zeta^{11} & \zeta^{12} \\ \zeta^{21} & \zeta^{22} \end{pmatrix} = \begin{pmatrix} \psi^0 + \psi^3 & \psi^1 - i\psi^2 \\ \psi^1 + i\psi^2 & \psi^0 - \psi^3 \end{pmatrix}, \quad (8)$$

where A, \dot{B} number rows and columns, respectively, and $\sigma^j, j = 1, 2, 3$, are the Pauli matrices, σ^0 is the unit matrix. For details of the spinor calculus reader should consult [3, 16, 17].

Equations (4) can be written within spinor formalism as:

$$\left. \begin{aligned} p^{A\dot{B}} \psi &= m \psi^{A\dot{B}} \\ p_{A\dot{B}} \psi^{A\dot{B}} &= 2m \psi \end{aligned} \right\}. \quad (9)$$

It follows from (9) that $p_{A\dot{B}} \psi^{A\dot{B}} = p_{A\dot{B}} p^{A\dot{B}} \psi$ and $p_{A\dot{B}} p^{A\dot{B}} \psi = 2m^2 \psi$. Moreover, $p_{A\dot{B}} p^{A\dot{B}} = p_{1\dot{1}} p^{1\dot{1}} + p_{2\dot{1}} p^{2\dot{1}} + p_{1\dot{2}} p^{1\dot{2}} + p_{2\dot{2}} p^{2\dot{2}} = 2p_\mu p^\mu$ and the Klein-Gordon equation $p_\mu p^\mu \psi = m^2 \psi$ follows. Let us note that due to spinor identities $p_{1\dot{1}} p^{1\dot{1}} + p_{2\dot{1}} p^{2\dot{1}} = p_\mu p^\mu$, $p_{1\dot{2}} p^{1\dot{2}} + p_{2\dot{2}} p^{2\dot{2}} = p_\mu p^\mu$ we can split the last of equations (9) and write Eqs.(9) as a set of two equations:

$$\left. \begin{aligned} p^{1\dot{1}} \psi &= m \psi^{1\dot{1}} \\ p^{2\dot{1}} \psi &= m \psi^{2\dot{1}} \\ p_{1\dot{1}} \psi^{1\dot{1}} + p_{2\dot{1}} \psi^{2\dot{1}} &= m \psi \end{aligned} \right\}, \quad (10)$$

$$\left. \begin{aligned} p^{1\dot{2}} \psi &= m \psi^{1\dot{2}} \\ p^{2\dot{2}} \psi &= m \psi^{2\dot{2}} \\ p_{1\dot{2}} \psi^{1\dot{2}} + p_{2\dot{2}} \psi^{2\dot{2}} &= m \psi \end{aligned} \right\}, \quad (11)$$

each of which describes particle with mass m (we check this substituting e.g. $\psi^{1\dot{1}}$, $\psi^{2\dot{1}}$ or $\psi^{1\dot{2}}$, $\psi^{2\dot{2}}$ into the third equations). Eq. (9) and the set of two equations (10), (11) are equivalent. We described equations (10), (11) in [11, 12]. From each of equations (10), (11) an identity follows:

$$p^{2\dot{1}}\psi^{1\dot{1}} = p^{1\dot{1}}\psi^{2\dot{1}}, \quad (12)$$

$$p^{2\dot{2}}\psi^{1\dot{2}} = p^{1\dot{2}}\psi^{2\dot{2}}. \quad (13)$$

Equation (10) and the identity (12), as well as equation (11) and the identity (13) can be written in form of the Dirac equations:

$$\begin{pmatrix} 0 & 0 & p^0 + p^3 & p^1 - ip^2 \\ 0 & 0 & p^1 + ip^2 & p^0 - p^3 \\ p^0 - p^3 & -p^1 + ip^2 & 0 & 0 \\ -p^1 - ip^2 & p^0 + p^3 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi^{1\dot{1}} \\ \psi^{2\dot{1}} \\ \chi \\ 0 \end{pmatrix} = m \begin{pmatrix} \psi^{1\dot{1}} \\ \psi^{2\dot{1}} \\ \chi \\ 0 \end{pmatrix}, \quad (14)$$

$$\begin{pmatrix} 0 & 0 & p^0 - p^3 & p^1 + ip^2 \\ 0 & 0 & p^1 - ip^2 & p^0 + p^3 \\ p^0 + p^3 & -p^1 - ip^2 & 0 & 0 \\ -p^1 + ip^2 & p^0 - p^3 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi^{2\dot{2}} \\ \psi^{1\dot{2}} \\ \chi \\ 0 \end{pmatrix} = m \begin{pmatrix} \psi^{2\dot{2}} \\ \psi^{1\dot{2}} \\ \chi \\ 0 \end{pmatrix}, \quad (15)$$

respectively, with one zero component, where explicit formulae for the spinor $p^{A\dot{B}}$ were used, cf. (8).

4 Subsolutions of the Dirac equation

4.1 Classical subsolutions of the Dirac equation

In the $m = 0$ case it is possible to obtain two independent equations for spinors ξ , η by application of projection operators $Q_{\pm} = \frac{1}{2}(1 \pm \gamma^5)$ to Eq.(1) since $\gamma^5 \stackrel{df}{=} -i\gamma^0\gamma^1\gamma^2\gamma^3$ anticommutes with $\gamma^\mu p_\mu$:

$$Q_{\pm}\gamma^\mu p_\mu \Psi = \gamma^\mu p_\mu (Q_{\mp}\Psi) = 0. \quad (16)$$

In the spinor representation of the Dirac matrices [3] we have $\gamma^5 = \text{diag}(-1, -1, 1, 1)$ and thus $Q_- \Psi = \begin{pmatrix} \xi \\ 0 \end{pmatrix}$, $Q_+ \Psi = \begin{pmatrix} 0 \\ \eta \end{pmatrix}$ and separate equations for ξ , η follow:

$$(p^0 + \vec{\sigma} \cdot \vec{p}) \eta = 0, \quad (17a)$$

$$(p^0 - \vec{\sigma} \cdot \vec{p}) \xi = 0, \quad (17b)$$

where $\vec{\sigma}$ denotes the vector built of the Pauli matrices. Equations (17) are known as the Weyl equations and are used to describe massless left-handed and right-handed neutrinos. However, since the experimentally established

phenomenon of neutrino oscillations requires non-zero neutrino masses, theory of massive neutrinos, which can be based on the Dirac equation, is necessary [19, 20, 18]. Alternatively, a modification of the Dirac or Weyl equation, called the Majorana equation, is thought to apply to neutrinos. According to Majorana theory neutrino and antineutrino are identical and neutral [21]. Although the Majorana equations can be introduced without any reference to the Dirac equation they are subsolutions of the Dirac equation [19].

Indeed, demanding in (1) that $\Psi = \mathcal{C}\Psi$ where \mathcal{C} is the charge conjugation operator, $\mathcal{C}\Psi = i\gamma^2\Psi^*$, we obtain in the spinor representation $\xi = -i\sigma^2\eta^*$, $\eta = i\sigma^2\xi^*$ and the Dirac equation (1) reduces to two separate Majorana equations for two-component spinors:

$$(p^0 + \vec{\sigma} \cdot \vec{p}) \eta = -im\sigma^2\eta^*, \quad (18a)$$

$$(p^0 - \vec{\sigma} \cdot \vec{p}) \xi = +im\sigma^2\xi^*. \quad (18b)$$

It follows from the condition $\Psi = \mathcal{C}\Psi$ that Majorana particle has zero charge built-in condition. The problem whether neutrinos are described by the Dirac equation or the Majorana equations is still open [19, 20, 18].

Let us note that the Dirac equations (1) in the spinor representation of the γ^μ matrices can be also separated in form of second-order equations:

$$(p^0 + \vec{\sigma} \cdot \vec{p}) (p^0 - \vec{\sigma} \cdot \vec{p}) \xi = m^2\xi, \quad (19)$$

$$(p^0 - \vec{\sigma} \cdot \vec{p}) (p^0 + \vec{\sigma} \cdot \vec{p}) \eta = m^2\eta. \quad (20)$$

Such equations were used by Feynman and Gell-Mann to describe weak decays in terms of two-component spinors [22].

4.2 Other massive subsolutions of the free Dirac equation

The free Dirac equation (1) in the spinor representation of γ matrices reads:

$$\left. \begin{aligned} (p^0 + p^3) \eta_1 + (p^1 - ip^2) \eta_2 &= m\xi^1 \\ (p^1 + ip^2) \eta_1 + (p^0 - p^3) \eta_2 &= m\xi^2 \\ (p^0 - p^3) \xi^1 + (-p^1 + ip^2) \xi^2 &= m\eta_1 \\ (-p^1 - ip^2) \xi^1 + (p^0 + p^3) \xi^2 &= m\eta_2 \end{aligned} \right\}, \quad (21)$$

with $\Psi = (\xi^1, \xi^2, \eta_1, \eta_2)^T$ [3] (see also [16, 17] for full exposition of spinor formalism).

In this Subsection we shall investigate other possibilities of finding subsolutions of the Dirac equation in the setting of first-order equations. For $m \neq 0$ we can define new quantities:

$$(p^0 + p^3) \eta_1 = m\xi_{(1)}^1, \quad (p^1 - ip^2) \eta_2 = m\xi_{(2)}^1, \quad (22a)$$

$$(p^1 + ip^2) \eta_1 = m\xi_{(1)}^2, \quad (p^0 - p^3) \eta_2 = m\xi_{(2)}^2, \quad (22b)$$

where we have:

$$\xi_{(1)}^1 + \xi_{(2)}^1 = \xi^1, \quad (23a)$$

$$\xi_{(1)}^2 + \xi_{(2)}^2 = \xi^2. \quad (23b)$$

In spinor notation $\xi_{(1)}^1 = \psi_1^{1\dot{1}}$, $\xi_{(2)}^1 = \psi_2^{1\dot{2}}$, $\xi_{(1)}^2 = \psi_1^{2\dot{1}}$, $\xi_{(2)}^2 = \psi_2^{2\dot{2}}$.

Equations (21) can be now written as

$$\left. \begin{aligned} (p^0 + p^3) \eta_{\dot{1}} &= m \xi_{(1)}^1 \\ (p^1 - ip^2) \eta_{\dot{2}} &= m \xi_{(2)}^1 \\ (p^1 + ip^2) \eta_{\dot{1}} &= m \xi_{(1)}^2 \\ (p^0 - p^3) \eta_{\dot{2}} &= m \xi_{(2)}^2 \\ (p^0 - p^3) (\xi_{(1)}^1 + \xi_{(2)}^1) + (-p^1 + ip^2) (\xi_{(1)}^2 + \xi_{(2)}^2) &= m \eta_{\dot{1}} \\ (-p^1 - ip^2) (\xi_{(1)}^1 + \xi_{(2)}^1) + (p^0 + p^3) (\xi_{(1)}^2 + \xi_{(2)}^2) &= m \eta_{\dot{2}} \end{aligned} \right\} \quad (24)$$

It follows from Eqs.(22) that the following identities hold:

$$(p^1 + ip^2) \xi_{(1)}^1 = (p^0 + p^3) \xi_{(1)}^2, \quad (25a)$$

$$(p^0 - p^3) \xi_{(2)}^1 = (p^1 - ip^2) \xi_{(2)}^2. \quad (25b)$$

Taking into account the identities (25) we can finally write equations (24) as a system of the following two equations:

$$\left. \begin{aligned} (p^0 + p^3) \eta_{\dot{1}} &= m \xi_{(1)}^1 \\ (p^1 + ip^2) \eta_{\dot{1}} &= m \xi_{(1)}^2 \\ (p^0 - p^3) \xi_{(1)}^1 + (-p^1 + ip^2) \xi_{(1)}^2 &= m \eta_{\dot{1}} \end{aligned} \right\}, \quad (26)$$

$$\left. \begin{aligned} (p^1 - ip^2) \eta_{\dot{2}} &= m \xi_{(2)}^1 \\ (p^0 - p^3) \eta_{\dot{2}} &= m \xi_{(2)}^2 \\ (-p^1 - ip^2) \xi_{(2)}^1 + (p^0 + p^3) \xi_{(2)}^2 &= m \eta_{\dot{2}} \end{aligned} \right\}. \quad (27)$$

Due to the identities (25) equations (26), (27) can be cast into form:

$$\left(\begin{array}{cccc} 0 & 0 & p^0 + p^3 & p^1 - ip^2 \\ 0 & 0 & p^1 + ip^2 & p^0 - p^3 \\ p^0 - p^3 & -p^1 + ip^2 & 0 & 0 \\ -p^1 - ip^2 & p^0 + p^3 & 0 & 0 \end{array} \right) \left(\begin{array}{c} \xi_{(1)}^1 \\ \xi_{(1)}^2 \\ \eta_{\dot{1}} \\ 0 \end{array} \right) = m \left(\begin{array}{c} \xi_{(1)}^1 \\ \xi_{(1)}^2 \\ \eta_{\dot{1}} \\ 0 \end{array} \right), \quad (28)$$

$$\left(\begin{array}{cccc} 0 & 0 & p^0 - p^3 & p^1 + ip^2 \\ 0 & 0 & p^1 - ip^2 & p^0 + p^3 \\ p^0 + p^3 & -p^1 - ip^2 & 0 & 0 \\ -p^1 + ip^2 & p^0 - p^3 & 0 & 0 \end{array} \right) \left(\begin{array}{c} \xi_{(2)}^1 \\ \xi_{(2)}^2 \\ \eta_{\dot{2}} \\ 0 \end{array} \right) = m \left(\begin{array}{c} \xi_{(2)}^1 \\ \xi_{(2)}^2 \\ \eta_{\dot{2}} \\ 0 \end{array} \right). \quad (29)$$

5 Supersymmetric equations and their symmetries

We shall now interpret the subsolutions equations (14), (15) and (28), (29)., First of all, we note that pairs of equations (14), (15) and (28), (29) are identical in form but have vector and spinor solutions, respectively. We shall thus refer to these equations as supersymmetric equations. We have demonstrated that equations (28) and (29) are Lorentz covariant [15] and that (14), (15) are charge conjugated one to another [14].

Let us consider Eqs.(14), (28). They can be written as:

$$\gamma^\mu p_\mu P_4 \Psi = m P_4 \Psi, \quad (30)$$

where P_4 is the projection operator, $P_4 = \text{diag}(1, 1, 1, 0)$ and spinor representation of the Dirac matrices. Incidentally, there are other projection operators which lead to analogous three component equations, $P_1 = \text{diag}(0, 1, 1, 1)$, $P_2 = \text{diag}(1, 0, 1, 1)$, $P_3 = \text{diag}(1, 1, 0, 1)$ but we shall need only the operator P_4 . Acting from the left on (30) with P_4 and $(1 - P_4)$ we obtain two equations:

$$P_4 (\gamma^\mu p_\mu) P_4 \Psi = m P_4 \Psi, \quad (31a)$$

$$(1 - P_4) (\gamma^\mu p_\mu) P_4 \Psi = 0. \quad (31b)$$

In the spinor representation of γ^μ matrices Eq.(31a) is equivalent to (26) while Eq.(31b) is equivalent to the identity (25a). Now the projection operator can be written as $P_4 = \frac{1}{4} (3 + \gamma^5 - \gamma^0 \gamma^3 + i \gamma^1 \gamma^2)$ (and similar formulae can be given for other projection operators P_1, P_2, P_3 , see [17] where another convention for γ^μ matrices was however used). It thus follows that the supersymmetric equation (30) is now given representation independent form.

6 Discussion

We have shown that subsolutions of the Dirac equation as well as of the DKP equations for spin 0 (similar subsolutions arise in the DKP theory for spin 1 [13]) obey the Dirac equation with built-in projection operator (30). Therefore, this covariant equation has bosonic as well as fermionic degrees of freedom and may provide a background for supersymmetric formalism. Let us note here that interaction can be incorporated into (30) via minimal action, $p^\mu \rightarrow \pi^\mu = p^\mu - e A^\mu$, but in the interacting case Eq.(30) is non-equivalent neither to the Dirac or the DKP equations [15].

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